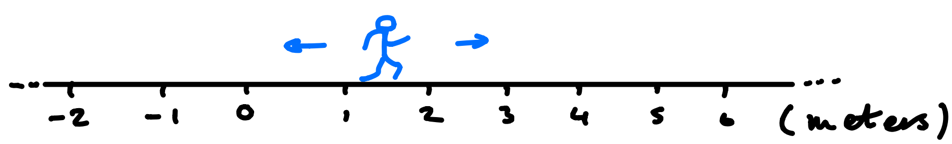
Rates of Change

Aim: Understand how a function changes as input changes.

Motivating Example: Motion in a straight line.



$s(t)$ = position at time t (in seconds)

Q: What is my average speed / velocity between $t = a$ and $t = b$?

$$\text{Average Speed over } [a,b] = \frac{\text{Displacement between } t=a \text{ and } t=b}{\text{Time elapsed between between } t=a \text{ and } t=b} = \frac{s(b)-s(a)}{b-a}$$

E.g.

t	0	1	2	3	4	5
$s(t)$	2	3	6	7	5	2

$\Rightarrow$

$$\text{Average Speed over } [1,3] = \frac{s(3)-s(1)}{3-1} = \frac{7-3}{3-1} = 2 \text{ m/s}$$

$$\text{Average Speed over } [2,5] = \frac{s(5)-s(2)}{5-2} = \frac{2-6}{5-2} = -\frac{4}{3} \text{ m/s}$$

Q: What is my speed/velocity at a single moment $t = a$?

Problem: No interval of time to measure displacement over.

Very Clever Solution: Look at average speed over smaller and smaller intervals containin $t=a$.

$$\begin{array}{l}\text{Instantaneous} \\ \text{Speed at } t=a\end{array} = \lim_{h \to 0} \begin{array}{l}\text{Average Speed} \\ \text{over } [a, a+h]\end{array} = \lim_{h\to 0} \frac{s(a+h) - s(a)}{h}$$

smaller and smaller interval containing $t=a$

Example $s(t) = t^2$. $\begin{array}{l}\text{Instantaneous} \\ \text{Speed at } t=1\end{array} = ?$

$$\begin{array}{l}\text{Instantaneous} \\ \text{speed at } t=1\end{array} = \lim_{h\to 0} \frac{(1+h)^2 - 1^2}{h}$$

$$\left.\begin{array}{l}\lim_{h\to 0} (1+h)^2 - 1^2 = (1+0)^2 - 1^2 = 0 \\ \lim_{h \to 0} h = 0\end{array}\right\} \Rightarrow \begin{array}{l}\text{Unclear} \\ \text{Quotient}\end{array}$$

$$\frac{(1+h)^2 - 1^2}{h} = \frac{\cancel{1^2} + 2h + h^2 - \cancel{1^2}}{h} = 2 + h$$

$$\Rightarrow \begin{array}{l}\text{Instantaneous} \\ \text{Speed at } t=1\end{array} = \lim_{h\to 0} 2+h = 2 \text{ m/s}.$$

Definition (Rate of Change) f - function

$$\frac{\text{Average rate of change of } f \text{ over } [a,b]}{} = \frac{f(b) - f(a)}{b - a}$$

$$\text{Instantaneous rate of change of } f \text{ at } x = a = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Remark: 1) This is precisely why we've introduced the concept of a limit. It allows us to understand how a function is changing at a single moment.

2) $C(x)$ = cost of producing x units
$R(x)$ = Revenue from selling x units
$P(x)$ = Profit from producing and selling x units

Instantaneous rate of change of $C/R/P$ at $x = a$ = Marginal Cost / Revenue / Profit at $x = a$

Example If the cost function is $C(x) = 2x^2 + x + 1$, calculate the marginal cost at $x = 4$.

Marginal Cost at $x=4$ $= \lim_{h \to 0} \dfrac{C(4+h) - C(4)}{h}$

$$= \lim_{h \to 0} \frac{(2(4+h)^2 + (4+h) + 1) - (2 \cdot 4^2 + 4 + 1)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2 \cdot 4^2} + 2 \cdot 8h + 2 \cdot h^2 + \cancel{4} + h + \cancel{1} - \cancel{2 \cdot 4^2} - \cancel{4} - \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{17 \cdot h + 2 \cdot h^2}{h}$$

polynomials are continuous

$$= \lim_{h \to 0} 17 + 2 \cdot h = 17 + 2 \cdot 0 = 17$$

Example (Harder) A population is increasing over time. Assume $P(t) = 1 + \sqrt{t}$ is the population (in millions) at time t (in years). What is the instantaneous rate of change at $t = 1$?

Instantaneous rate of change at $t=1$ $= \lim_{h \to 0} \dfrac{P(1+h) - P(1)}{h}$

$$= \lim_{h \to 0} \frac{(1 + \sqrt{1+h}) - (1 + \sqrt{1})}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+h} - \sqrt{1}}{h}$$

Unclear Quotient

$$= \lim_{h \to 0} \frac{(\sqrt{1+h} - \sqrt{1})(\sqrt{1+h} + \sqrt{1})}{h(\sqrt{1+h} + \sqrt{1})}$$ ← Algebra trick

$$= \lim_{h \to 0} \frac{(1+h) - 1}{h(\sqrt{1+h} + \sqrt{1})} = \lim_{h \to 0} \frac{h}{h(\sqrt{1+h} + \sqrt{1})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{1+h} + \sqrt{1}}$$

$$\lim_{h \to 0} 1 = 1$$

function is continuous

$$\lim_{h \to 0} \sqrt{1+h} + \sqrt{1} = \sqrt{1+0} + \sqrt{1} = 2$$

$\Rightarrow$ Instantaneous Rate of change at $t=1$ $= \frac{1}{2}$ million/year